OPTIMAL CONTROL

Topics Covered

- State-feedback control
- Linear Quadratic Regular (LQR) design
- Bryson's Rule

Prerequisites

- Hardware Interfacing laboratory experiment.
- Filtering laboratory experiment.
- State-Space Representation laboratory experiment.

1 Background

Sometimes, we would like to design a controller that provides the *best* results with respect to some *performance measure* or, alternatively, a controller that minimizes a certain cost function. For instance, let us assume that we want to control the trajectory of a vehicle. We would like to keep the error between the actual and desired states of the vehicle small while using very little control (e.g., fuel). In that case, we aim to design a feedback controller that stabilizes the system and drives the state error to zero while using the least possible control effort. The optimal *Linear Quadratic Regulator* (LQR) helps us to achieve this goal.

1.1 Linear State-Space Model

The standard state-space representation of a multi-input multi-output (MIMO) continuous linear-time invariant (LTI) system with n state variables, r input variables, and m output variables is

$$\dot{x}(t) = Ax + Bu \tag{1.1}$$

and

$$y(t) = Cx(t) + Du(t) \tag{1.2}$$

where x is the vector of state variables $(n \times 1)$, u is the control input vector $(r \times 1)$, y is the output vector $(m \times 1)$, A is the system matrix $(n \times n)$, B is the input matrix $(n \times r)$, C is the output matrix $(m \times n)$, and D is the feed-forward matrix $(m \times r)$.

For the DC motor, the state and output are defined

$$x = \begin{bmatrix} \theta_m(t) & \dot{\theta}_m(t) \end{bmatrix}^T \tag{1.3}$$

and

$$y = \theta_m. \tag{1.4}$$

Remark that only the position of the DC motor is being measured directly and, as a result, only θ_m is included in the output equation.

1.2 Controllability

If the system is unstable, we might be able to design a state-feedback controller to stabilize it. Even if the system is stable, we may still want to regulate the performance of the system according to some design specifications (e.g., final state, rate of convergence, and settling time). These are possible if the system is *controllable*. By controllable, we mean that for any initial state vector x_a and any desired final state x_b, there exists a control input u that can steer the state of the system from x_a to x_b in finite time [1]. Otherwise, we say that the system is *uncontrollable*. Note that the definition does not require u to be bounded.

To check if the system is controllable, we can compute the rank of the *Controllability* matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{1.5}$$

where n is the dimension of the state vector x. Then, we say that the linear system given in Equation 1.1 or, equivalently, the pair (A, B) is controllable if and only if rank$\{\mathcal{C}\} = n$. Otherwise, we conclude that the system, or the pair (A, B), is uncontrollable.

1.3 State-Feedback Control

The state-feedback control of a system is shown in Figure 1.1. Let u be a state feedback control law of the form

$$u = -Kx \tag{1.6}$$

where $K \in \Re^{m \times n}$ is called the feedback gain (note that for the DC-Motor, there is only one control input and, therefore, $m = 1$). The closed-loop system equation is found by applying state-feedback to Equation 1.1

$$\dot{x} = Ax - BKx = (A - BK)x. \tag{1.7}$$

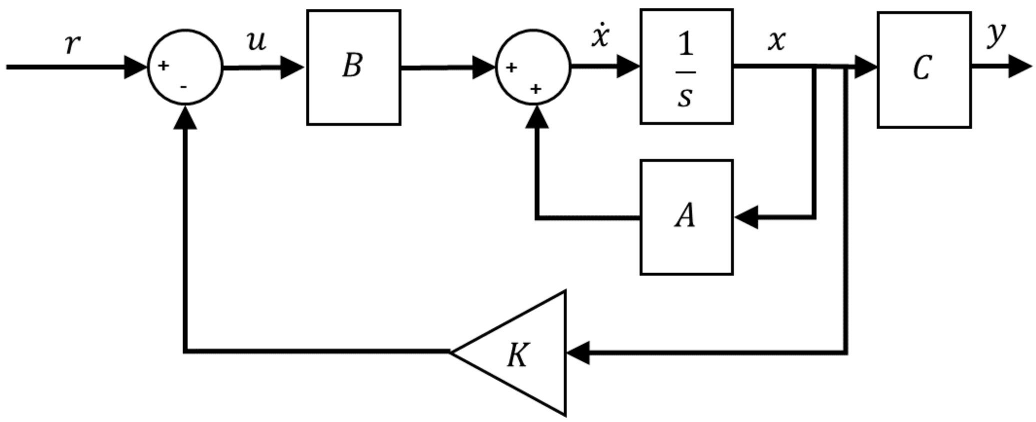

Figure 1.1: State-feedback block diagram

1.4 Optimal Control (LQR)

The Linear Quadratic Regular (LQR) design process consists of finding a state feedback control law that minimizes a cost function given by

$$J = \int_0^\infty \left(x^\top Q x + u^\top R u\right) \tag{1.8}$$

where $Q \in \Re^{n \times n}$ and $R \in \Re^{m \times m}$ are symmetric, positive-definite matrices specifying the minimization criteria.

Applying the state-feedback control in Equation 1.6 to the cost function in Equation 1.8, we can show that the control law that minimizes J is

$$K = R^{-1} B^\top P \tag{1.9}$$

where P is a positive-definite matrix found by solving the following algebraic Riccati equation

$$A^\top P + PA - PBR^{-1}B^\top P + Q = 0. \tag{1.10}$$

In the LQR method, Q and R are design parameters that *weight* the importance we place on the state error and control input, respectively. Increasing the values of Q will place more emphasis on minimizing the state vector and increase the overall control effort u and the corresponding control gain K. Increasing the value of R will *decrease* the overall control effort required and generate a lower overall control gain K.

One method to choose Q and R is *Bryson's Rule* [2]. Bryson's rule suggests the use of diagonal matrices Q and R with diagonal entries given by

$$\begin{aligned} Q_{ii} &= \frac{1}{\text{maximum acceptable value of } x_i^2}, & i \in \{1, \cdots, n\} \\ R_{jj} &= \frac{1}{\text{maximum acceptable value of } u_j^2}, & j \in \{1, \cdots, m\}. \end{aligned} \tag{1.11}$$

This method can at least act as a good starting point in selecting the Q and R matrices, but additional tuning may be necessary.

2 In-Lab Exercises

The Simulink model shown in Figure 2.1 implements a state-feedback control on the QUBE-Servo 2 to control the position of the DC motor.

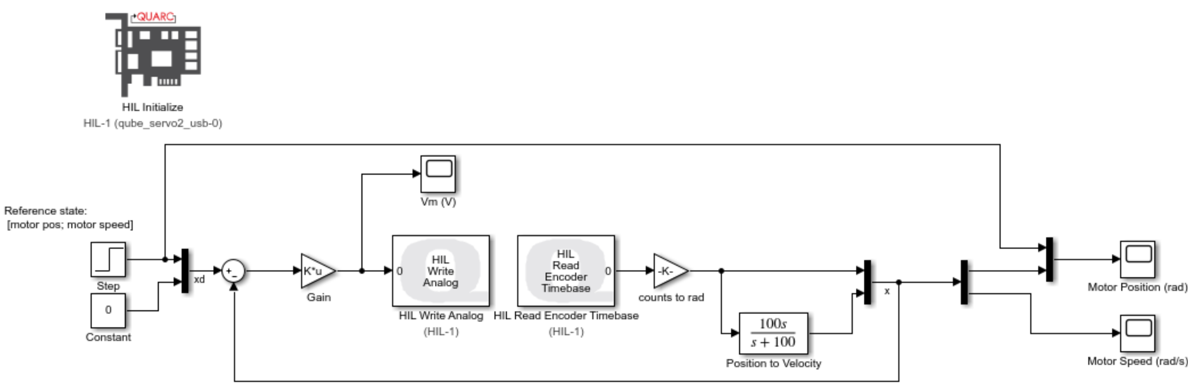

Figure 2.1: Simulink/QUARC model implementing state-feedback control on QUBE-Servo 2

2.1 LQR Control Design

In this section, Bryson's Rule will be used to find the Q and R matrices and the control gain K will be found mathematically.

1. Use Bryson's Rule to find weighting matrices Q and R that satisfy the following maximum DC motor position and velocity as well as motor input voltage limit:

$$\begin{aligned} \theta_{max} &= 1 \text{ rad} \\ \omega_{max} &= 119.0 \text{ rad/s} \\ V_{max} &= 5 \text{ V} \end{aligned} \tag{2.1}$$

   Note: The amplifier limit of the QUBE-Servo 2 is ± 10 V. However, in order to prevent actuator saturation we limit the control effort to 5V. The maximum angular rate is based on this limitation such that $5/k_m = 5/0.042 = 119$ rad/s.

2. Evaluating the DC motor state-space model found in State-Space Representation laboratory experiment using the model parameters

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -10.05 \end{bmatrix} s + \begin{bmatrix} 0 \\ 239.4 \end{bmatrix} u \tag{2.2}$$

3. Using the state-space model, solve the Riccati equation in order to find the three equations that will be used (in the next exercise) to find the P matrix terms: p_{11}, $p_{12} = p_{21}$, and p_{22}. The P terms will be then be used to find the control gain.

   Hint: $P \in \Re^{2\times 2}$ is a real symmetric matrix. Therefore the diagonal terms $p_{12} = p_{21}$.

4. Given the solved Riccati equation, find the control gain using Equation 1.9. Use the Q and R found in the Step 1.

   Hint: Because this is two-state single-input system and looking at Equation 1.9, only the p_{12} and p_{22} terms need to be solved. Also, because P is real symmetric matrix matrix all its terms need to be positive.

5. Open MATLAB®.
6. Similarly as done in State-Space Representation laboratory experiment, create a MATLAB® script that constructs a MATLAB state-space model using the parameters defined in the the MATLAB® script `qube2_param.m` provided.
7. Use the MATLAB `LQR` command to generate the control gain using the weighting matrices found in 2.1. Attach the commands used in the MATLAB® script and the resulting control gain generated, K.
8. How does the gain found in MATLAB® compare with the gain found manually through calculation in Step 4?

2.2 DC Motor Optimal Control

1. Create the Simulink model shown in Figure 2.1 using the model designed in Filtering laboratory experiment or open the q_qube2_lqr Simulink model (if supplied).
2. Run the script designed in Section 2.1 to generate the control gain K.
3. Build and run the model in QUARC. A sample response shown in Figure 2.2 shows the input motor voltage and the measured position and velocity response of the QUBE-Servo 2 using an example control gain of

$$K = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

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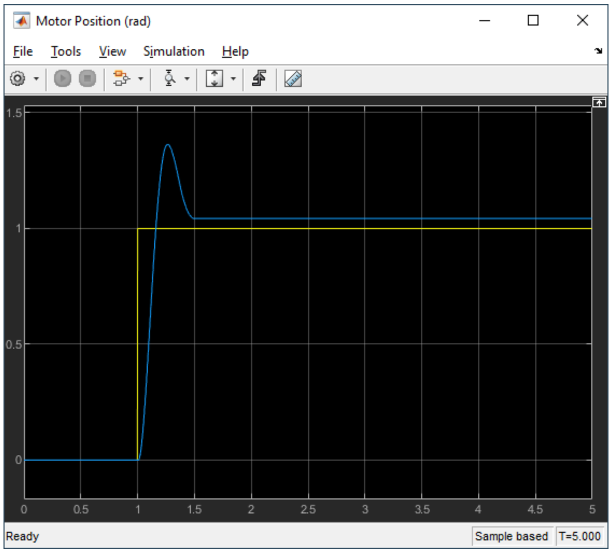
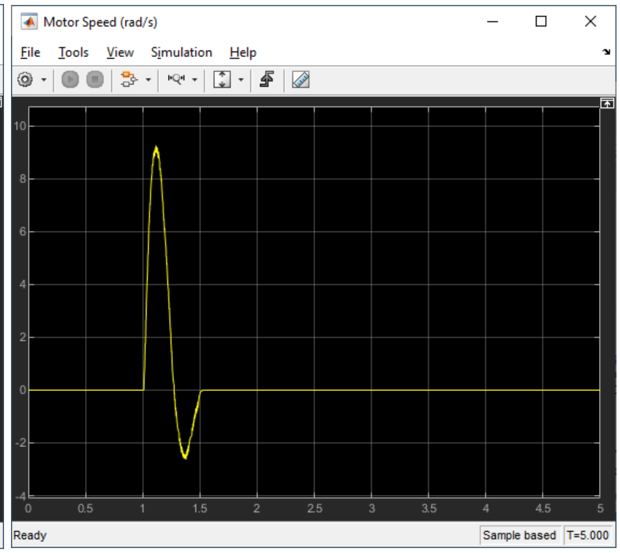
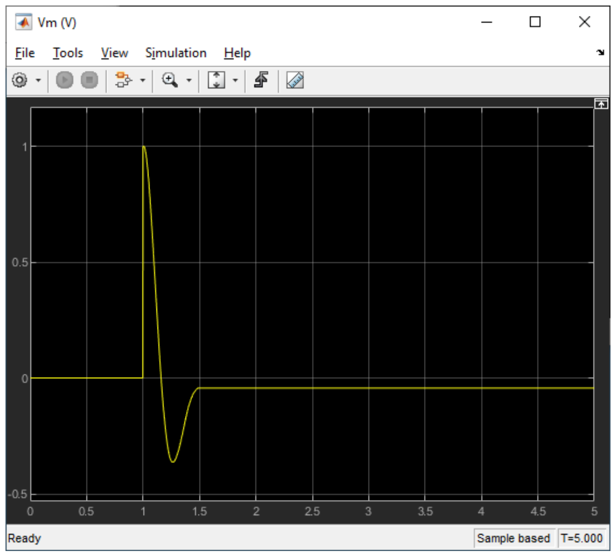


(a) Position



(b) Velocity



(c) Voltage

Figure 2.2: Response using an example state-feedback gain.

4. Attach the response you obtained when the LQR gain you designed is used. Does this satisfy the specifications given in Equation 2.1?

5. Stop the QUARC controller and power off the QUBE-Servo 2 if no more experiments will be performed.

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[2] Gene F. Franklin, J. David Powell, and Abbas Emami-Naeini. *Feedback Control Of Dynamic Systems*. Prentice Hall, 2006.