DISCRETE STABILITY

Topics Covered

- Assess stability of discrete system from pole locations in z-domain
- Discrete root-locus design of a proportional controller

Prerequisites

- Hardware Interfacing laboratory experiment.
- PD Control laboratory experiment.
- Introduction to Discrete Control laboratory experiment.

1 Background

For continuous time system design, the analysis of the system and controller design is often carried out in the frequency domain. The Laplace transform is used to convert the time-domain signal to the complex s-domain. The Laplace Transform of a continuous time-domain signal $f(t)$ is

$$\mathcal{L}\left[f(t)\right] = F(s) = \int_0^\infty f(t)e^{-st}dt. \tag{1.1}$$

The continuous system $F(s)$ is defined as being *stable* if all its poles are in the left half plane of the complex s-plane.

For discrete time system design analysis, a similar approach is used. Here, the sampled time signal $f(nT)$ of the continuous signal $f(t)$ is used for the z-transform

$$\mathcal{Z}\left\{f(k)\right\} = F(z) = \sum_0^\infty f(k)z^{-k}, \tag{1.2}$$

where T is the sample time of the discrete system and $n = 0, 1, 2, 3, \ldots$ are the signal samples. The relationship between the s-plane and z-plane with respect to equivalent characteristics of the system is given by the relationship

$$z = e^{sT}, \tag{1.3}$$

where T is the sampling period of the discrete system.

The stability of a discrete control system depends on the locations of its closed-loop system poles on the z-plane, as shown in Figure 1.1. A discrete system is:

1. Stable if all the poles are inside the unit circle.
2. Unstable if any pole lies outside the unit circle and/or there are more than one pole on the unit circle.
3. Marginally stable if one pole lies on the unit circle and all other poles are inside the unit circle.

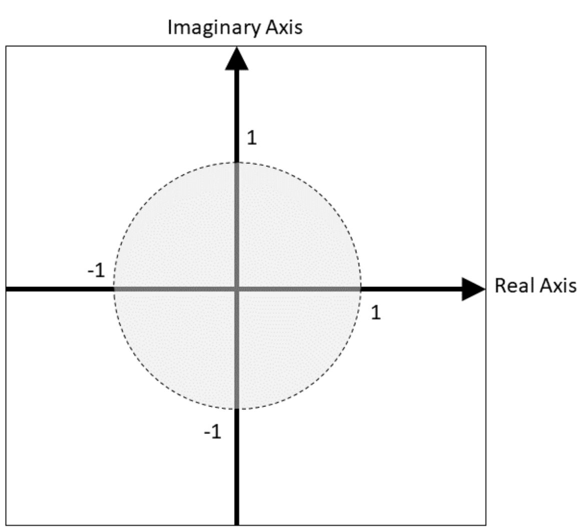


Figure 1.1: Stability on the z-plane

The unity feedback block diagram with the sample and zero-order hold that model the digital computer is shown in Figure 1.2.

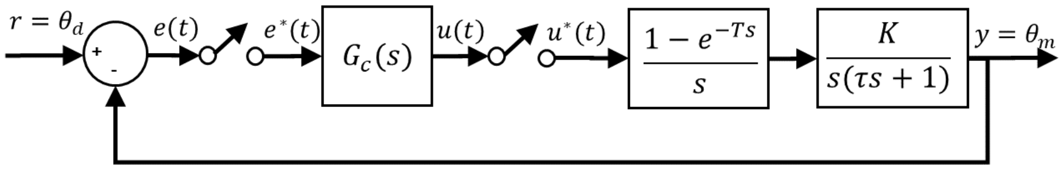


Figure 1.2: Closed-loop system with compensator implemented on digital computer

Recall the QUBE-Servo 2 voltage-to-position transfer function is

$$P(s) = \frac{\Theta_m(s)}{V_m(s)} = \frac{K}{s(\tau s + 1)}, \tag{1.4}$$

where K is the model steady-state gain, τ is the model time constant, $\Theta_m(s) = \mathcal{L}\left[\theta_m(t)\right]$ is the motor/disk position, and $V_m(s) = \mathcal{L}\left[v_m(t)\right]$ is the applied motor voltage.

With the sample and hold, this becomes:

$$G_p(s) = \frac{1 - e^{-Ts}}{s} \frac{K}{s(\tau s + 1)} \tag{1.5}$$

The additional integrator in the s-domain expression is based on the step input from the zero-order hold during each sampling period. In the discrete domain, a single step with a unit duration is equivalent to a step of infinite duration combined with the same step of opposite sign one time step delayed, i.e. $1 - z^{-1}$. Thus taking the z-transform of $G_p(s)$ gives

$$G_p(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{K}{s(\tau s + 1)}\right] = (1 - z^{-1})\mathcal{Z}\left[\frac{P(s)}{s}\right] \tag{1.6}$$

The closed-loop transfer function between the reference input, r, and the system output, y, in the block diagram in Figure 1.2 is

$$T(z) = \frac{Y(z)}{R(z)} = \frac{G_c(z)G_p(z)}{1 + G_c(z)G_p(z)} \tag{1.7}$$

and the corresponding closed-loop error signal is

$$E(z) = \frac{R(z)}{1 + G_c(z)G_p(z)} \tag{1.8}$$

2 In-Lab Exercises

2.1 Stability Analysis

1. Find the z-transform of open-loop system with the sample and hold, $G_p(z)$.
2. Assess the stability of the open-loop system by looking at the poles in the z-plane. Is the stability affected by the sampling interval, T? If so, explain.
3. Design a MATLAB script that models the discrete open-loop transfer function and plots the pole-zero map. Model the discrete system using different sampling intervals, e.g. $T = 0.03$ s and $T = 0.002$ s, and examine how this affects the pole-zero locations. Does the sampling affect the poles as expected?

   Hint: Use the MATLAB® `c2d` command with the 'zoh' option to convert a continuous-time model into discrete using the zero-order hold method.

2.2 Discrete Root-Locus Control Design

The Simulink model shown in Figure 2.1 implements the discrete unity feedback control shown in Figure 1.2.

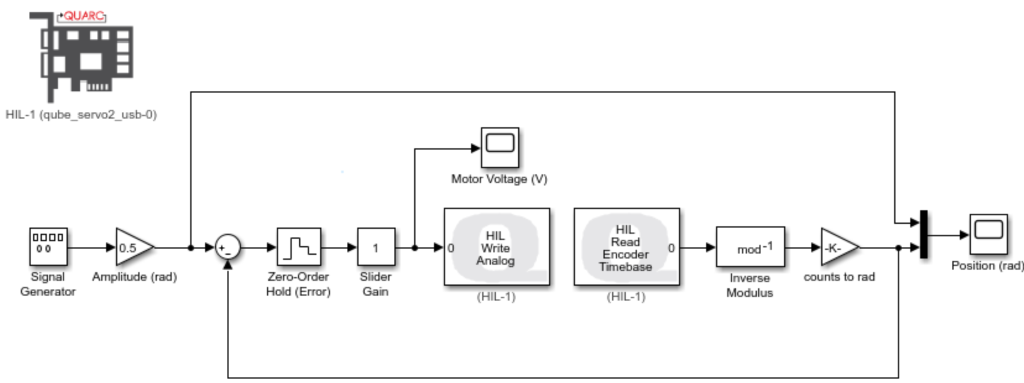
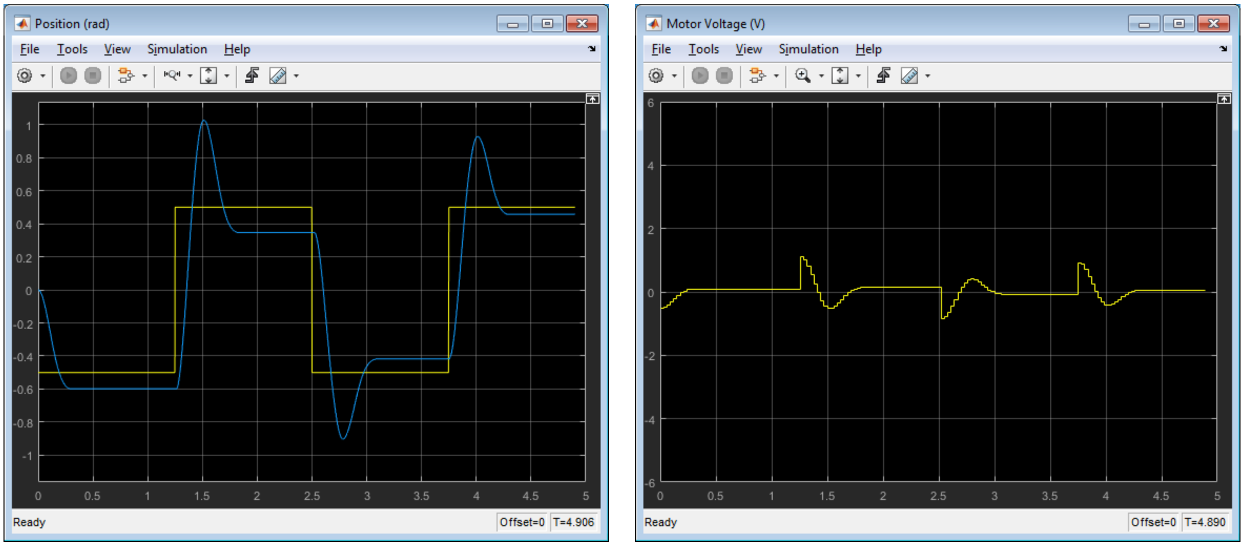


Figure 2.1: Simulink/QUARC model implementing a discrete unity feedback system

1. Open the MATLAB®.
2. The MATLAB® *Control System Designer* tool is used for system stability analysis and control design. Open the *Control System Designer* by entering `controlSystemDesigner` in the MATLAB command prompt to see how the proportional gain affects the system when it is sampled at 33.3 Hz. Can the closed-loop poles be moved inside the unit circle? If so, for what range of k_p is the system stable? Attach representative screen captures showing the root-locus and step response plots from *Control System Designer* to support your answer.

   Hint: Use the command `controlSystemDesigner(Gz)` where `Gz` is the discretized open-loop transfer function $G(z)$ defined in Equation 1.5 at sampling interval of 0.03 s.
3. Connect and power up the QUBE-Servo 2 system and ensure the inertial disc load is mounted.
4. Design (or open, if supplied) the Simulink model shown in Figure 2.1.
5. Set the Signal Generator block such that the servo command (i.e. reference angle) is a square wave with an amplitude of 0.5 rad and at a frequency of 0.4 Hz.
6. Set the sampling interval of the Zero-Order Hold to 0.03 s by entering `Ts = 0.03` in the MATLAB prompt.
7. Build and run the QUARC® controller. The response should look similarly as shown in Figure 2.2 for a proportional gain of 1.



(a) Position

(b) Voltage

Figure 2.2: QUBE-Servo 2 response with $k_p = 1$ V/rad at discrete sampling rate of $T = 0.03$ s

8. When running the proportional controller on the QUBE-Servo 2 at 33.3 Hz, how does the system behave when the gain is lowered and when it is increased? Is this in-line with the analysis performed using *Control System Designer*? Attach responses from the scopes to support your answer.

9. Stop the QUARC controller.

10. Power off the QUBE-Servo 2 if no other experiments will be conducted.

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