PENDULUM STATE SPACE MODELING

Topics Covered

- First-principles modeling
- State-space representation
- Model validation

Prerequisites

- Hardware Interfacing laboratory experiment.
- Rotary Pendulum Modeling laboratory experiment.

1 Background

1.1 Rotary Pendulum Model

The rotary pendulum model is shown in Figure 1.1. The rotary arm pivot is attached to the QUBE-Servo 2 system and is actuated. The arm has a length of r, a moment of inertia of J_r, and its angle θ increases positively when it rotates counter-clockwise (CCW). The servo (and thus the arm) should turn in the CCW direction when the control voltage is positive, $v_m > 0$.

The pendulum link is connected to the end of the rotary arm. It has a total length of L_p and it center of mass is at $l = L_p/2$. The moment of inertia about its center of mass is J_p. The rotary pendulum angle α is zero when it is hanging downward and increases positively when rotated CCW.

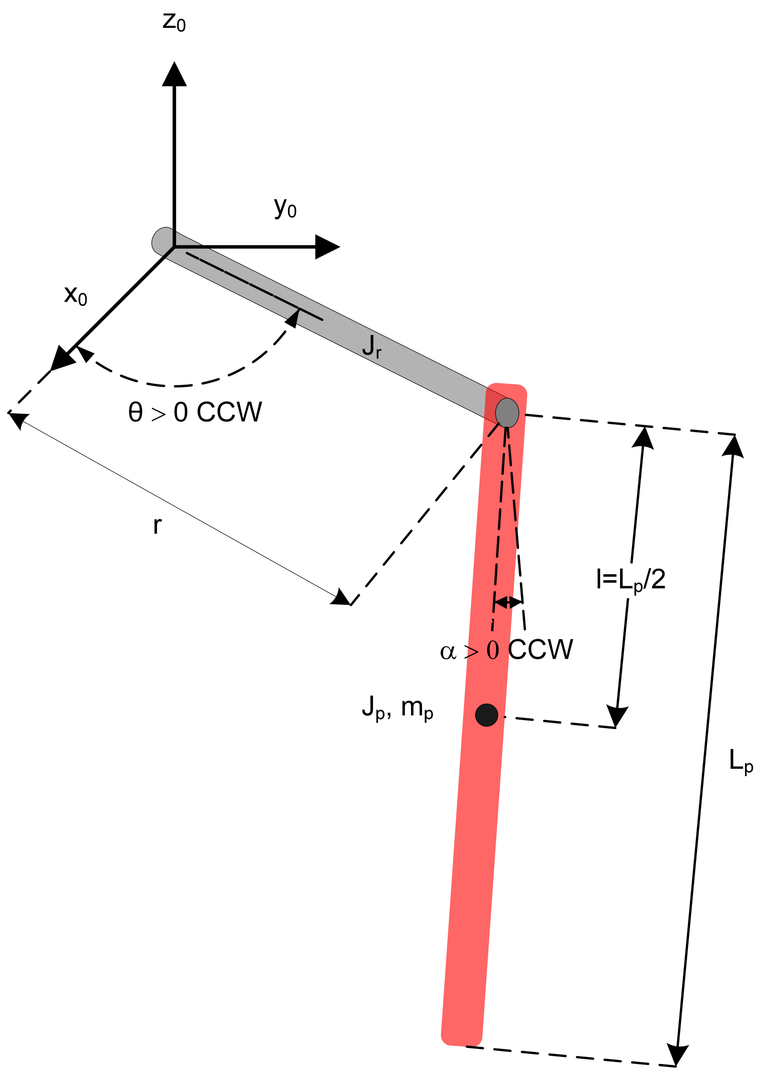


Figure 1.1: Rotary pendulum model

The equations of motion (EOM) for the pendulum system were developed using the Euler-Lagrange method. This systematic method is often used to model complicated systems such as robot manipulators with multiple joints. The total kinetic and potential energy of the system is obtained, then the Lagrangian can be found. A number of derivatives are then computed to yield the EOMs. The resultant nonlinear EOM are:

$$\left(J_r + J_p \sin\alpha^2\right)\ddot{\theta} + m_p l r \cos\alpha\ddot{\alpha} + 2J_p \sin\alpha\cos\alpha\dot{\theta}\dot{\alpha} - m_p l r \sin\alpha\dot{\alpha}^2 = \tau - b_r\dot{\theta} \tag{1.1}$$

and

$$J_p\ddot{\alpha} + m_p l r \cos\alpha\ddot{\theta} - J_p \sin\alpha\cos\alpha\dot{\theta}^2 + m_p g l \sin\alpha = -b_p\dot{\alpha}. \tag{1.2}$$

where $J_r = m_r r^2/3$ is the moment of inertia of the rotary arm with respect to the pivot (i.e. rotary arm axis of rotation) and $J_p = m_p L_p^2/3$ is the moment of inertia of the pendulum link relative to the pendulum pivot (i.e. axis of rotation of pendulum). The viscous damping acting on the rotary arm and the pendulum link are b_r and b_p, respectively. The applied torque at the base of the rotary arm generated by the servo motor is

$$\tau = \frac{k_m}{R_m}\left(v_m - k_m\dot{\theta}\right) \tag{1.3}$$

1.2 Linear Model

When the nonlinear EOM are linearized about the operating point, the resultant linear EOM for the rotary pendulum are defined as:

$$J_r\ddot{\theta} + m_p l r\ddot{\alpha} = \tau - b_r\dot{\theta} \tag{1.4}$$

and

$$J_p\ddot{\alpha} + m_p l r\ddot{\theta} + m_p g l\alpha = -b_p\dot{\alpha}. \tag{1.5}$$

Solving for the acceleration terms yields:

$$\ddot{\theta} = \frac{1}{J_t}\left(m_p^2 l^2 r g\alpha - J_p b_r\dot{\theta} + m_p l r b_p\dot{\alpha} + J_p\tau\right) \tag{1.6}$$

and

$$\ddot{\alpha} = \frac{1}{J_t}\left(-m_p g l J_r\alpha + m_p l r b_r\dot{\theta} - J_p b_p\dot{\alpha} - m_p r l\tau\right). \tag{1.7}$$

where

$$J_t = J_p J_r - m_p^2 l^2 r^2. \tag{1.8}$$

1.3 Linear State-Space Model

The linear state-space equations are

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.9}$$

and

$$y(t) = Cx(t) + Du(t) \tag{1.10}$$

where x is the vector of state variables $(n \times 1)$, u is the control input vector $(r \times 1)$, y is the output vector $(m \times 1)$, A is the system matrix $(n \times n)$, B is the input matrix $(n \times r)$, C is the output matrix $(m \times n)$, and D is the feed-forward matrix $(m \times r)$.

The block diagram representation of state-space equations is shown in Figure 1.2.

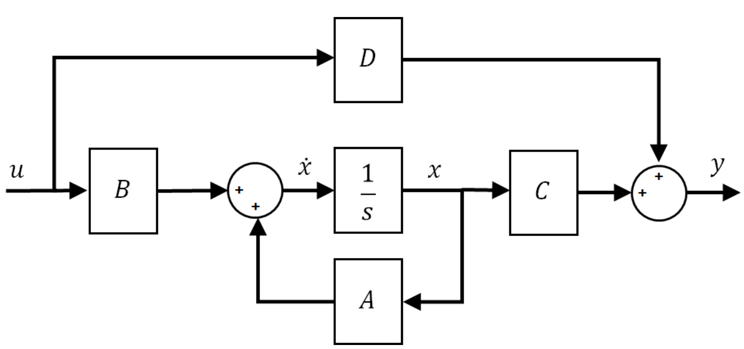


Figure 1.2: Block diagram of state-space system

For the rotary pendulum system, the state and output are defined

$$x(t) = \begin{bmatrix} \theta(t) & \alpha(t) & \dot{\theta}(t) & \dot{\alpha}(t) \end{bmatrix}^T \tag{1.11}$$

and

$$y(t) = \begin{bmatrix} \theta(t) & \alpha(t) \end{bmatrix}^T . \tag{1.12}$$

The system state x defines the state variables needed to model the system and output state y defines the state variables that are measured directly.

2 In-Lab Exercises

2.1 Pendulum State-Space Model

1. Based on the output state defined above, find the state-space matrices C and D in Equation 1.10. Why are there only two variables defined in the output equation?

2. Using Equation 1.6 and Equation 1.7 and the defined state in Equation 1.11, derive the linear state-space model of the pendulum system.

3. In MATLAB®, write a script that models the rotary pendulum using the state-space representation matrices derived in Step 2. Use the model parameters defined in the `qube2_rotpen_param.m` script supplied (i.e. make sure you use the same variable names) and the `rotpen_ABCD_eqns_down_student.m` as a starting point.

   Note: The A, B, C, and D matrices need to be completed in `rotpen_ABCD_eqns_down_student.m`. Define the matrices as done in Step 2, when the control input is defined as torque, i.e. $u = \tau$. The last few lines of the scripts will convert the model to be with respect to motor voltage, i.e. $u = v_m$.

2.2 Model Validation

The SIMULINK® model below runs for 5 sec and applies a step voltage input to the QUBE-Servo 2 and its state-space model. The output scopes display the responses of the rotary arm and pendulum angle of the QUBE-Servo 2 (yellow) in parallel with the response from the linear model of the system (blue).

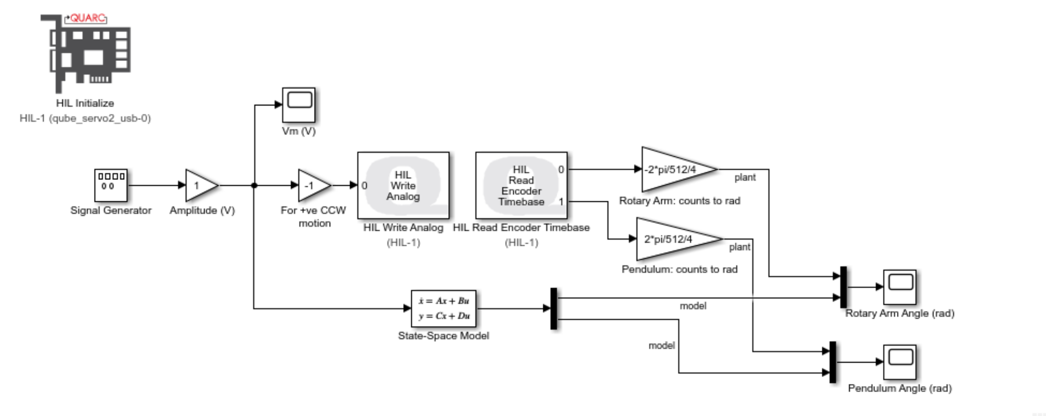


Figure 2.1: Applies a step voltage and displays measured and simulated pendulum response.

1. Open (if supplied) or design the SIMULINK® model shown in Figure 2.1 that applies a 1 V, 1 Hz square wave to the pendulum system and state-space model.

   Hint: If you are designing the Simulink diagram, use the model already developed in the Rotary Pendulum Modeling laboratory experiment.

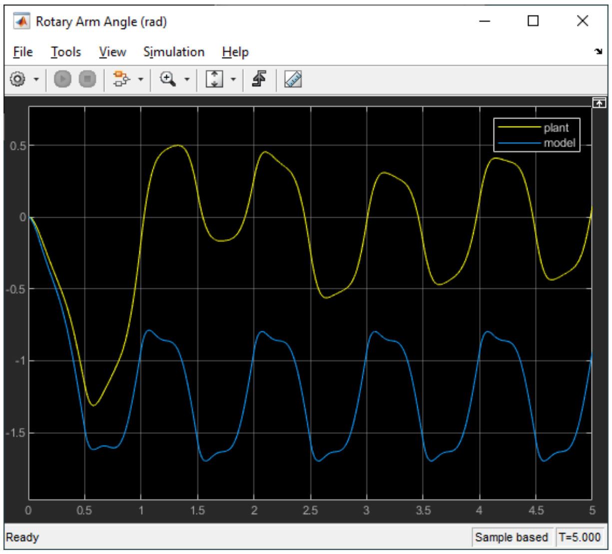
2. Run `setup_rotpen_ss_model.m` to create the state space model parameters in the MATLAB® workspace. Ensure that the generated matrices match your solution in Step 2.

3. In `qube2_rotpen_param.m`, set the rotary arm viscous damping coefficient b_r to 0.001 N.m.s/rad, and the pendulum damping coefficient b_p to 0.00005 N.m.s/rad. These parameters were found experimentally to reasonably accurately reflect the viscous damping of the system due to effects such as friction, when subject to a step response.

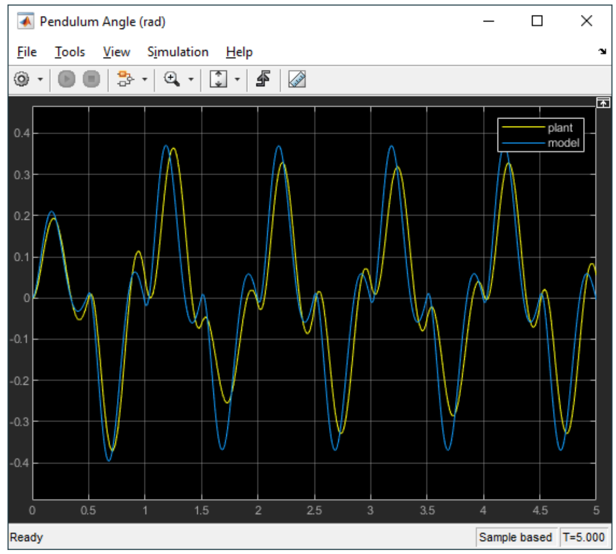
4. Build and run the QUARC controller.

5. The scope response should be similar to Figure 2.2. Attach the rotary pendulum response - showing both the measured and simulated (model based) rotary arm and pendulm angles.

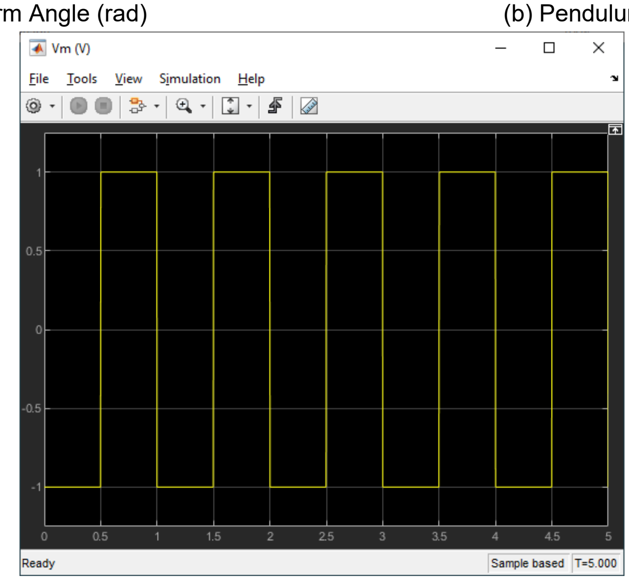
   Hint: For information on saving data to MATLAB® for offline analysis, see the QUARC® help documentation (under *QUARC Targets* | *User's Guide* | *QUARC Basics* | *Data Collection*). You can then use the MATLAB® `plot` command to generate the necessary MATLAB figure.



(a) Rotary Arm Angle (rad)



(b) Pendulum Angle (rad)



(c) Motor Voltage (V)

Figure 2.2: Step response of the rotary pendulum system

6. Does your model represent the actual pendulum well? If not, explain why there might be discrepancies.

7. The viscous damping of each pendulum can vary slightly from system to system. If your model does not accurately represent your specific pendulum system, try modifying the damping coefficients b_r and b_p to obtain a more accurate model.

8. Stop the QUARC® controller and power off the QUBE-Servo 2 if no more experiments will be conducted.

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