ROUTH-HURWITZ COEFFICIENT TEST

Topics Covered

- System stability
- Closed-loop transfer function
- Routh-Hurwitz Coefficient Test.

Prerequisites

- Integration laboratory experiment.
- Filtering laboratory experiment.
- Second-Order Systems laboratory experiment.

1 Background

To determine the stability of a system, it is necessary to investigate whether or not all closed-loop poles are in the left half of the s-plane. If there is one or more poles in the right half plane, the system is unstable. If there is a pole on the imaginary axis, the system is marginally stable. If and only if all poles are strictly in the left half plane the system is said to be stable. Especially for higher order systems, it is not always feasible to find the exact locations of all system poles to determine the overall stability of the system. Instead, a classification of how many poles are in each portion of the s-plane is sufficient.

1.1 Routh-Hurwitz Criterion

The Routh-Hurwitz Criterion is a two step process to determine the system's stability *without* having to obtain exact pole locations. In the first step, a Routh table is created, which is then interpreted in the second step to find out how many poles are in the left half plane, on the imaginary axis, or in the right half plane. The major benefit of the Routh-Hurwitz Criterion is its inherent capability to determine the range for which unknown system parameters result in a stable closed-loop system response.

The closed-loop transfer function for a generic feedback loop as shown in Figure 1.1 is given as

$$\frac{Y(s)}{R(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{N(s)}{D(s)}. \tag{1.1}$$

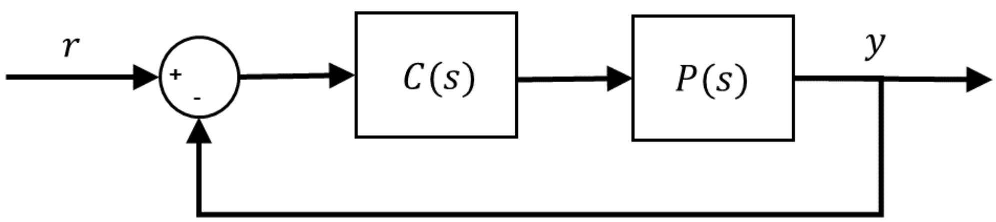


Figure 1.1: Closed-loop feedback loop

To determine the stability of Equation 1.1, we are only interested in the pole locations of the closed-loop system, i.e. the roots of the closed-loop denominator, where $D(s)$ has the form

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2}^{n-2} + \ldots + a_0 s^0. \tag{1.2}$$

If and only if all poles are strictly in the left half plane, the system is stable.

For example, a fourth-order polynomial has the following form

$$D(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0. \tag{1.3}$$

To create a Routh Table, we add a row for each coefficient of $D(s)$ and add, starting from the highest coefficient, every other coefficient in decreasing order in the first line. Next, repeat the procedure for the second line, starting with the second highest coefficient and add every other coefficient in decreasing order similar to the first line. The Routh Table for Equation 1.3 looks like Table 1.1.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Table 1.1: Enter the polynomial coefficients in the Routh Table

The remaining entries are found as illustrated in Table 1.2 for the system given in Equation 1.3. Each entry is a negative determinant of a 2×2 matrix divided by the value in the first column one row above. The entries in the first column of the determinant are the 2 values of the rows above. The entries in the second column of the determinant are the values in the 2 rows above in the column to the right of the current entry.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Table 1.2: Completed Routh table

Once the Routh Table is found, the number of sign changes in the first column is equal to the number of right half plane (unstable) system poles.

Case 1: Row is all zeros

If an entire row in the Routh Table is zero, the denominator polynomial has a factor that is an even polynomial. To complete the Routh Table for this case, move up one row, write out the corresponding polynomial of that row using the entries of the row as the coefficients. Then differentiate this polynomial and use the coefficients of the differentiated polynomial instead of the zeros previously obtained. For example, consider the polynomial

$$s^5 + 7s^4 + 5s^3 + 35s^2 + 2s + 14. \tag{1.4}$$

The corresponding Routh Table is

s^5	1	5	2
s^4	7	35	14
s^3	0	0	0
s^2			
s^1			
s^0			

The s^3 row has all zero coefficients, thus the polynomial with coefficients one row above is $p(s) = 7s^4 + 35s^2 + 14$. Differentiating $p(s)$ yields

$$\frac{dp(s)}{ds} = 28s^3 + 70s + 0. \tag{1.5}$$

Using these coefficients instead of the zero row above and finding the remaining entries as outlined above yields the completed Routh Table

s^5	1	5	2
s^4	5	35	14
s^3	28	70	0
s^2	22.5	14	0
s^1	52.58	0	0
s^0	14	0	0

Every even polynomial only has roots that are symmetrical about the origin. Since there is no sign change in the first column of the Routh Table, there are no positive roots. Therefore, the only possible symmetry is for the roots to be purely imaginary, thus the overall system is *marginally stable*.

Case 2: Zero in the First Column

It may also happen that there is only a leading zero in the Routh Table and the remaining entries in that row are non-zero. In that case, replace the leading zero with ϵ and carry on deriving the remainder of the Routh Table as a function of ϵ. Once the whole table is obtained, determine the sign changes and stability with $\epsilon > 0$ and $\lim_{\epsilon \to 0}$. Consult the literature for more details.

2 In-Lab Exercises

In this lab, you'll investigate the phenomenon of marginally stability using the Routh-Hurwitz criterion. In order to obtain a third-order transfer function, a unity feedback with a simple compensator as shown in Figure 2.1 will be implemented on the QUBE-Servo 2.

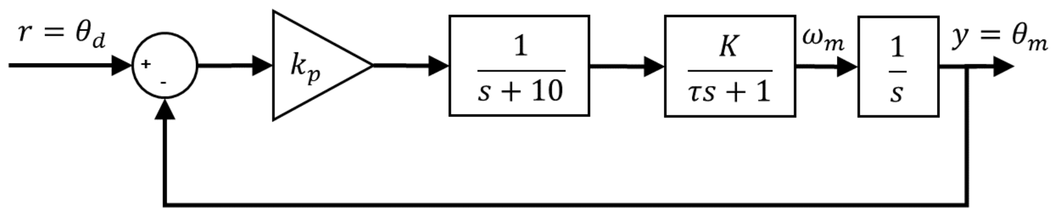


Figure 2.1: Unity feedback loop with compensator and proportional gain on Quanser Controls Board

Based on Figure 2.1, we have the compensator

$$C(s) = \frac{k_p}{s+10} \tag{2.1}$$

where k_p is the proportional gain. The plant voltage to position transfer function model of the DC motor

$$P(s) = \frac{1}{s(\tau s+1)} \tag{2.2}$$

where the steady-state gain $K = 23.2$ V/rad and the time constant $\tau = 0.13$ s.

Note: Note that there is also a proportional gain in the forward path of the feedback loop.

2.1 Routh Stability Analysis

1. Given the generic feedback loop diagram in Figure 1.1 where $P(s) = H(s)G(s)$ and $C(s) = k_p$, determine the closed-loop transfer function from $R(s)$ to $Y(s)$ in terms of k_p.
2. Create a Routh Table for the closed-loop transfer function.
3. Find the range of k_p for which the system is stable.
4. Is there a k_p for which the system is marginally stable? Comment on the pole locations of the closed-loop transfer function.

2.2 Testing Stability

In order to test the stability of the system shown in Figure 2.1, the SIMULINK® model shown in Figure 2.2 will be used.

1. Open the `q_qube2_routh_hurwitz` model shown in Figure 2.2 (if supplied) or build it using the model designed in previous labs, e.g. Second-Order Systems laboratory experiment.
2. Set the Amplitude (rad) block to 0 to apply a motor position reference signal of zero (i.e. zero setpoint).
3. Set the Proportional Gain to 1.

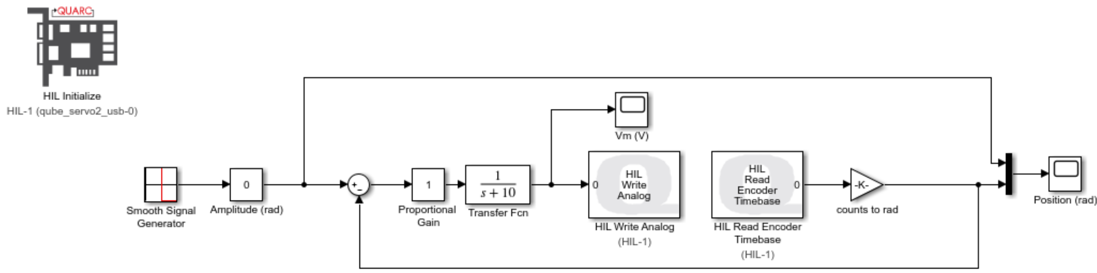
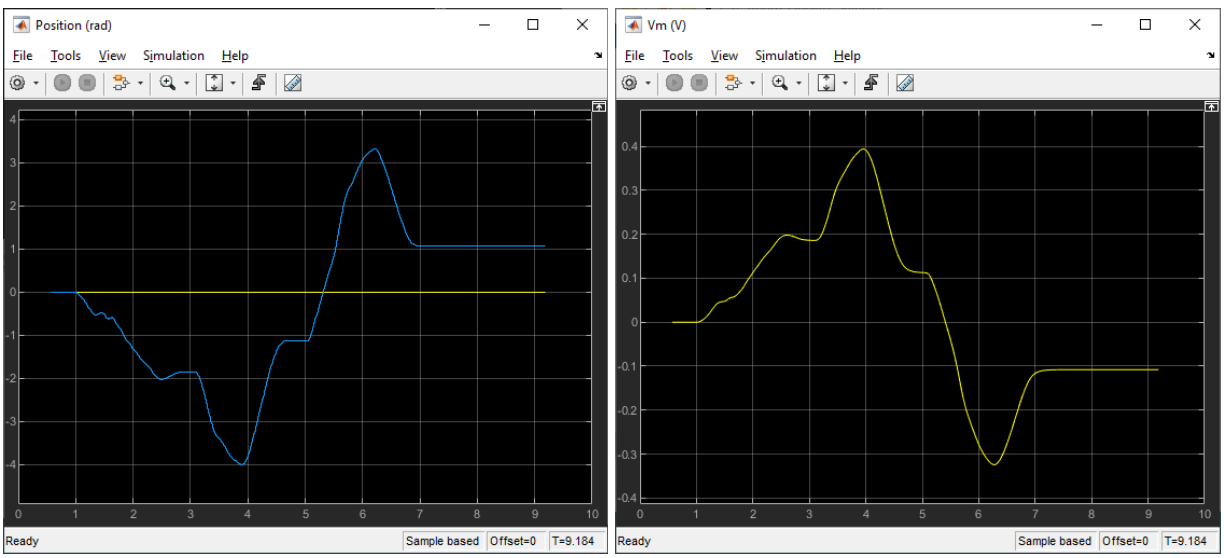


Figure 2.2: Simulink/QUARC model used to test stability of QUBE-Servo 2

4. Build and run the QUARC® controller.
5. While the controller is running, describe what happens when you manually perturb the load disk on the QUBE-Servo 2 and attach a representative response. An example response is shown in Figure 2.3. Is the response stable? Is there a steady state error once you have perturbed the system? If so, explain.



(a) Motor position (b) Applied voltage

Figure 2.3: Example response when perturbing the disk with $k_p = 1$

6. While the QUARC® controller is running, slowing increase the proportional gain to a value of about 5 V$/$rad. What effect does this have on the system response to a disturbance? Compare the steady state error to the one with a proportional gain of $k_p = 1$ and discuss any changes.
7. Increase the proportional gain until you reach the value for k_p for which the system is theoretically only marginally stable. How does the response compare to before? Attach a response showing you results.
8. Stop the QUARC® controller and power off the QUBE-Servo 2.

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