STEADY-STATE ERROR

Topics Covered

- Evaluating steady-state error.
- System types.
- Motor position control.
- Proportional-integral-derivative (PID) control.

Prerequisites

- Hardware Interfacing laboratory experiment.
- Filtering laboratory experiment.
- PD Control laboratory experiment.

1 Background

1.1 Steady-State Error

The block diagram shown in Figure 1.1 is a general unity feedback system with controller $C(s)$ and plant transfer function $P(s)$. The measured output, $Y(s)$, is supposed to track the reference signal $R(s)$.

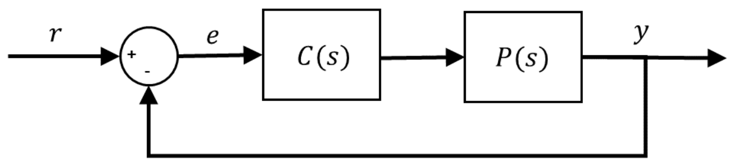

Figure 1.1: Block diagram of closed-loop system with plant model $P(s$ and controller $C(s)$

Steady-state error is illustrated in Figure 1.2 when the control system is attempting to track a ramp input signal and is denoted by the variable e_{ss}. It is the difference between the reference input and output signals after the system response has settled. Thus, for a time t when the system is in steady-state, the steady-state error equals

$$e_{ss} = r_{ss}(t) - y_{ss}(t) \tag{1.1}$$

where $r_{ss}(t)$ is the steady-state value of the reference input signal and y_{ss} is the steady-state value of the output signal.

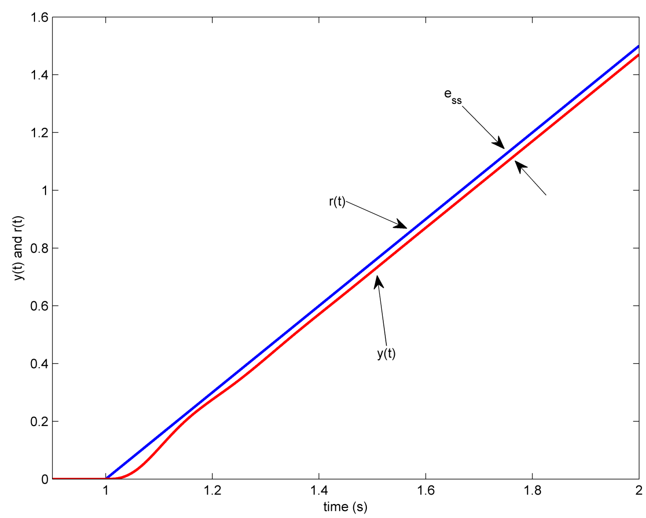

Figure 1.2: Steady-state error in ramp response.

The error transfer function $E(s)$ is then given by the Laplace transform of the error in Equation 1.1 as

$$E(s) = R(s) - Y(s). \tag{1.2}$$

From Figure 1.1, it can be shown that the reference to error closed-loop transfer function is

$$E(s) = \frac{1}{1 + P(s)C(s)} R(s) \tag{1.3}$$

where $C(s)$ is the controller and $P(s)$ is the plant transfer function.

1.1.1 Final Value Theorem

The Final Value Theorem can be used to determine the steady-state or final value of the system output $y(t)$ given its Laplace transform $Y(s)$. For a system with an unstable pole (i.e. a pole in the right half of the s-plane), the final value is unbounded. If the system is marginally stable with a complex conjugate pole pair on the imaginary axis, the output of the system will be oscillatory and the final value is not defined. For a stable system response (i.e. all poles of the system are strictly in the left half of the s-plane), the following holds

$$y_{ss}(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) \tag{1.4}$$

where $Y(s) = \mathcal{L}\left[y(t)\right]$ is the Laplace Transform of $y(t)$.

1.1.2 System Types

The *System Type* classifies systems according to the type of input signal it can track. In particular, it relates to the degree of the polynomial of the input signal.

Using the Final Value Theorem (assuming $sE(s)$ is stable), the steady-state error of the system due to a step input is

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{1+P(s)C(s)} = \frac{1}{1+\lim_{s\to 0} P(s)C(s)} = \frac{1}{1+K_p} \tag{1.5}$$

where $K_p = \lim_{s\to 0} P(s)C(s)$.

When a ramp input $R(s) = 1/s^2$ is applied, the steady-state error is

$$e_{ss} = \lim_{s\to 0} \frac{1}{s+sP(s)C(s)} = \frac{1}{\lim_{s\to 0} sP(s)C(s)} = \frac{1}{K_v} \tag{1.6}$$

where $K_v = \lim_{s\to 0} sP(s)C(s)$.

Due to parabolic input $R(s) = 1/s^3$, the steady-state error

$$e_{ss} = \lim_{s\to 0} \frac{1}{s^2+s^2P(s)C(s)} = \frac{1}{\lim_{s\to 0} s^2P(s)C(s)} = \frac{1}{K_a} \tag{1.7}$$

where $K_a = \lim_{s\to 0} s^2P(s)C(s)$.

The type of signal can be summarized by Table 1.1.

Type of Input	Step	Ramp	Parabolic
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

Table 1.1: System Types

1.2 PID Control

The Proportional-Integral-Derivative (PID) controller shown in Figure 1.3 will be used to control the position of the QUBE-Servo 2.

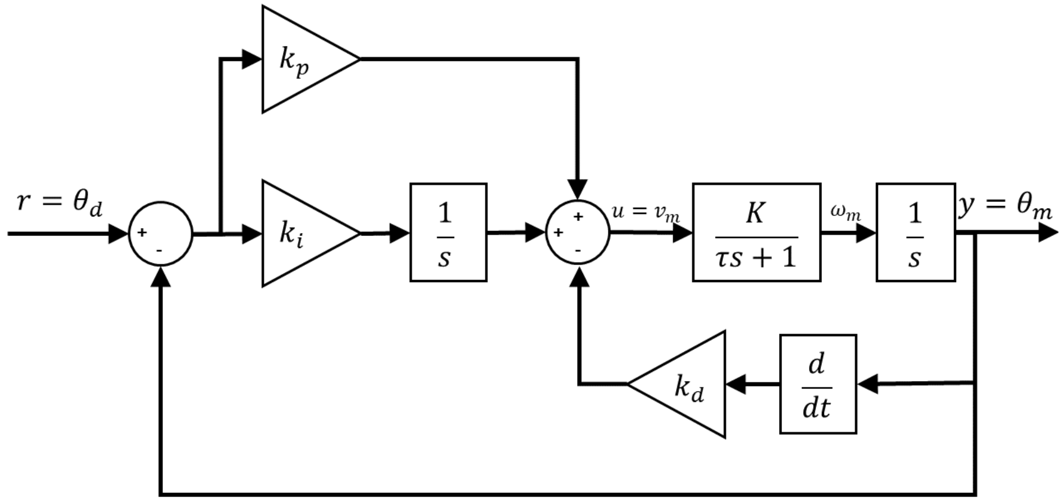

Figure 1.3: Block diagram of PID servo position control

Note that this is a variation of the standard PID control that feeds back the derivative of the servo position (i.e. not the derivative of the error like conventional PID control).

The time-domain equation of this PID control is

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau - k_d \frac{dy(t)}{dt}, \tag{1.8}$$

with corresponding transfer function

$$U(s) = \left(k_p + \frac{k_i}{s}\right)(R(s) - Y(s)) - k_d Y(s). \tag{1.9}$$

The voltage-to-position transfer function of the QUBE-Servo 2 is given as

$$P(s) = \frac{\Theta(s)}{V_m(s)} = \frac{K}{(\tau s + 1)\, s}, \tag{1.10}$$

where K is the model steady-state gain, τ is the model time constant, $\Omega_m(s) = \mathcal{L}\left[\omega_m(t)\right]$ is the load disk position, and $V_m(s) = \mathcal{L}\left[v_m(t)\right]$ is the applied motor voltage. The default values for the model parameters are $K = 22.4$ rad/s/V and $\tau = 0.15$ s.

Substituting Equation 1.9 into Equation 1.10 and solving for $Y(s)/R(s)$ gives the voltage-to-position closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{\frac{Kk_p}{\tau} s + \frac{Kk_i}{\tau}}{s^3 + \frac{1+Kk_d}{\tau} s^2 + \frac{Kk_p}{\tau} s + \frac{Kk_i}{\tau}}. \tag{1.11}$$

2 In-Lab Exercises

In this lab, we will investigate the steady-state error of a step and ramp reference signal when controlling the position using PD and PID controllers. Based on the model designed in PD Control laboratory experiment, build the Simulink diagram shown in Figure 2.1 that implements a full PID control, as shown in Figure 1.3.

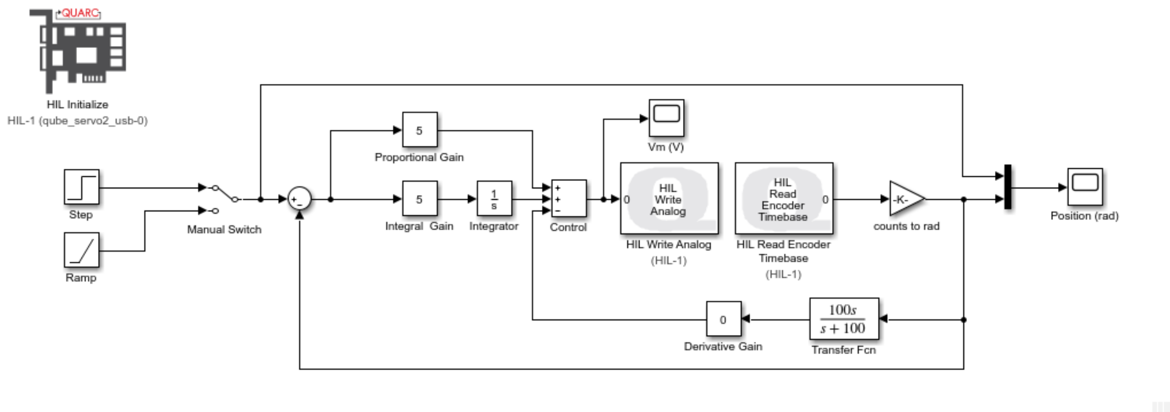

Figure 2.1: Simulink/QUARC model implementing PID position control with a ramp reference

2.1 Steady-State Error due to Step Input

1. Find the closed-loop transfer function $E(s)/R(s)$ in Figure 1.3 that represents the dynamics between the desired position, $R(s) = \Theta_d(s)$, and the error, $E(s) = Y(s) - R(s) = \Theta_m(s) - \Theta_d(s)$.
2. Find the error transfer function $E(s)$ for the step reference input
$$R(s) = \frac{R_0}{s}. \tag{2.1}$$
where R_0 is the step magnitude.
3. Evaluate the steady-state error of the system due to a step input when using a PD controller (i.e. $k_i = 0$). What is the system type?
4. Open the `q_qube2_ess` Simulink model shown in Figure 2.1 (if supplied) or design your own.
5. Set the proportional, integral, and derivative gains to $k_p = 5$ V/rad, $k_d = 0.25$ V/(rad/s), and $k_i = 0$. This will implement the PD only control.
6. Set the Manual Switch block to the Step signal.
7. Set the Amplitude of the Step block to 1. This will generate a step reference of 1 rad. The model duration is set to stop at 5.0 sec.
8. Build and run the QUARC controller.
9. Attach the step position response. What is the steady-state error of the system?. Compare the steady-state error obtained experimentally with your calculation in Step 3. Do they match? If not, give one reason why there is a difference.

2.2 Steady-State Error due to Ramp Input

1. Find the error transfer function $E(s)$ for the ramp reference input
$$R(s) = \frac{R_0}{s^2}.$$
where R_0 is the slope magnitude.

2. Find the steady-state error of the system due to a ramp input when using a PD controller (i.e. $k_i = 0$) and the PID controller. Compare and comment on the results.
3. Compute the expected steady-state error of the system using a PD controller when applying a ramp input with an amplitude of $R_0 = 5$ rad/s. Use the either the default model parameters given above or, preferably, use more precise model parameters found by going through one of the modeling experiments (e.g. Step Response Modeling laboratory experiment), and the PD gains $k_p = 5$ V/rad and $k_d = 0.25$ V/(rad/s).
4. Based on your analysis, what is the System Type for the PD and PID control systems?
5. Open the `q_qube2_ess` Simulink model shown in Figure 2.1 (if supplied) or design your own.
6. Set the proportional, integral, and derivative gains to $k_p = 5$ V/rad, $k_d = 0.25$ V/(rad/s), and $k_i = 0$. This will implement the PD only control.
7. Set the Manual Switch block to the Ramp signal.
8. Set the Amplitude of the ramp reference block to 5. This will make the slope of the ramp go up 5 rad/s. The model duration is set to stop at 5.0 sec.
9. Build and run the QUARC controller.
10. Examine the position response in the scope. Compare the steady-state error obtained with your analysis in Step 3. Do they match? If not, give one reason why there is a difference. Attach the ramp response.
11. Set the integral gain to $k_i = 5$ V/rad/s. Run the QUARC controller again. Does the response behave as you expected? Attached the ramp response and explain your results.
12. How could the position tracking of the ramp be improved? Show any experimental results and explain what you did.
13. Make sure the QUARC controller is stopped. Power off the QUBE-Servo 2 if no additional experiments will be performed.
14. Stop the QUARC® controller and turn off the QUBE-Servo 2.

© 2020 Quanser Inc., All rights reserved.

Quanser Inc.
119 Spy Court
Markham, Ontario
L3R 5H6
Canada
info@quanser.com
Phone: 1-905-940-3575
Fax: 1-905-940-3576

Printed in Markham, Ontario.

For more information on the solutions Quanser Inc. offers, please visit the web site at:
http://www.quanser.com

This document and the software described in it are provided subject to a license agreement. Neither the software nor this document may be used or copied except as specified under the terms of that license agreement. Quanser Inc. grants the following rights: a) The right to reproduce the work, to incorporate the work into one or more collections, and to reproduce the work as incorporated in the collections, b) to create and reproduce adaptations provided reasonable steps are taken to clearly identify the changes that were made to the original work, c) to distribute and publically perform the work including as incorporated in collections, and d) to distribute and publicly perform adaptations. The above rights may be exercised in all media and formats whether now known or hereafter devised. These rights are granted subject to and limited by the following restrictions: a) You may not exercise any of the rights granted to You in above in any manner that is primarily intended for or directed toward commercial advantage or private monetary compensation, and b) You must keep intact all copyright notices for the Work and provide the name Quanser Inc. for attribution. These restrictions may not be waved without express prior written permission of Quanser Inc.