POLE-PLACEMENT CONTROL

Topics Covered

- Controllability
- State-feedback control
- Companion matrices.
- Pole-placement control design
- Control design for higher order systems

Prerequisites

- Filtering laboratory experiment.
- Stability Analysis laboratory experiment.
- Rotary Pendulum Modeling laboratory experiment.

1 Background

1.1 State-Feedback Control

The standard state-space representation of a multi-input multi-output (MIMO) continuous linear-time invariant (LTI) system with n state variables, r input variables, and m output variables is

$$\dot{x}(t) = Ax + Bu \tag{1.1}$$

and

$$y(t) = Cx(t) + Du(t) \tag{1.2}$$

where $x \in \Re^{n\times 1}$ is the vector of state variables, $u \in \Re^{r\times 1}$ is the control input vector, $y \in \Re^{m\times 1}$ is the output vector, $A \in \Re^{n\times n}$ is the system matrix, $B \in \Re^{n\times r}$ is the input matrix, $C \in \Re^{m\times n}$ is the output matrix, and $D \in \Re^{m\times r}$ is the feed-forward matrix.

The state-feedback control of a system is shown in Figure 1.1. Let u be a state feedback control law of the form

$$u = -Kx \tag{1.3}$$

where $K \in \Re^{m\times n}$ is the feedback gain. Applying this to Equation 1.1 gives the closed-loop system

$$\dot{x} = Ax - BKx = (A - BK)x. \tag{1.4}$$

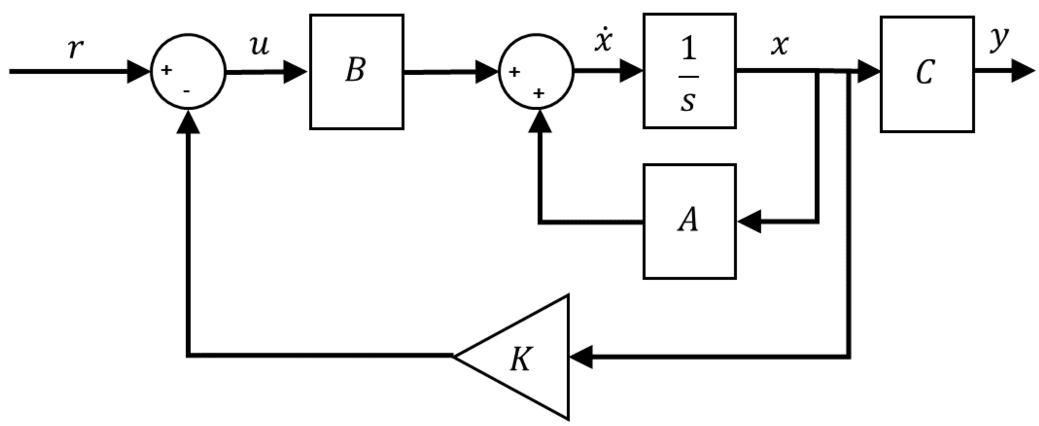


Figure 1.1: State-feedback block diagram

The state vector x of the rotary pendulum system is defined

$$x = \begin{bmatrix} \theta & \alpha & \dot{\theta} & \dot{\alpha} \end{bmatrix}^T. \tag{1.5}$$

The reference signal includes the desired rotary arm position, θ_r and is defined

$$x_{ref} = \begin{bmatrix} \theta_r & 0 & 0 & 0 \end{bmatrix}^T \tag{1.6}$$

and the control strategy used to balance the pendulum (and track the rotary arm setpoint) is

$$u = K(x_{ref} - x) = k_{p,\theta}(\theta_r - \theta) - k_{p,\alpha}\alpha - k_{d,\theta}\dot{\theta} - k_{d,\alpha}\dot{\alpha}. \tag{1.7}$$

1.2 Controllability

If the system is unstable, we might be able to design a state-feedback controller to stabilize it. Even if the system is stable, we may still want to regulate the performance of the system according to some design specifications (e.g., final state, rate of convergence, and settling time). These are possible if the system is *controllable*. By controllable, we mean that for any initial state vector x_a and any desired final state x_b, there exists a control input u that can steer the state of the system from x_a to x_b in finite time [1]. Otherwise, we say that the system is uncontrollable. Note that the definition does not require u to be bounded.

To check if the system is controllable, we can compute the rank of the *Controllability* matrix

$$\mathcal{C} = \begin{bmatrix} B \; AB \; A^2B \; \cdots A^{n-1}B \end{bmatrix} \tag{1.8}$$

where n is the dimension of the state vector x. Then, we say that the linear system Equation 1.1 or, equivalently, the pair (A, B) is controllable if and only if rank$\{\mathcal{C}\} = n$. Otherwise, we conclude that the linearized system (or, in other words, the pair (A, B)) is uncontrollable.

1.3 Pole Placement

If (A, B) is controllable, we can use the state-feedback to place the poles at desired locations, e.g. in the left half of the s-plane. Thus we can find a gain K for the closed-loop system in Equation 1.4 to obtain a desired characteristic equation

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s^1 + a_0 = 0$$

where $a_i \in \Re$ for $i \in \{0, 1, \cdots, n-1\}$.

To illustrate how to design K to obtain desired pole location, i.e. desired characteristic equation, consider the following example:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -1 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The characteristic equation of the open loop system is

$$|sI - A| = s^3 + 5s^2 + s - 3 = 0.$$

Its open loop poles are at $\{-4.64, -1, 0.65\}$, therefore the system is unstable. To obtain the closed-loop poles $\{-1, -2, -3\}$, the *desired* characteristic equation is

$$(s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6 \tag{1.9}$$

Applying the state-feedback control $u = -Kx$ with the gain $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$ the closed-loop system becomes

$$A - KB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 - k_1 & -1 - k_2 & -5 - k_3 \end{bmatrix}$$

The characteristic equation of $A - KB$ is

$$s^3 + (k_3 + 5)s^2 + (k_2 + 1)s + (k_1 - 3) \tag{1.10}$$

Equating the coefficients of Equation 1.10 with the desired polynomial in Equation 1.9

$$k_1 - 3 = 6$$
$$k_2 + 1 = 11$$
$$k_3 + 5 = 6$$

Solving for the gains, we find that a gain of $K = \begin{bmatrix} 9 & 10 & 1 \end{bmatrix}$ is required to move the poles to their desired location.

1.3.1 Companion Matrix

If (A, B) is controllable and a single-input system, i.e. $B \in \Re^{n \times 1}$, then A is similar to a companion matrix. Let the characteristic equation of A be

$$s^n + a_n s^{n-1} + \cdots + a_2 s^1 + a_1 = 0$$

where $a_i \in \Re$ for $i \in \{1, 2, \cdots, n\}$. Then the companion matrices of A and B are

$$\tilde{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \end{bmatrix} \tag{1.11}$$

and

$$\tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{1.12}$$

The companion matrices can be found using the matrix transformation

$$W = T\tilde{T}^{-1}$$

where T is the controllability matrix defined in Equation 1.8 and

$$\tilde{T} = \begin{bmatrix} \tilde{B} & \tilde{B}\tilde{A} & \ldots & \tilde{B}\tilde{A}^n \end{bmatrix}.$$

Then

$$\tilde{A} = W^{-1}AW$$

and

$$\tilde{B} = W^{-1}B.$$

1.3.2 Design Procedure Summary

The procedure to design a gain K for a controllable (A, B) system can be generalized as follows:

Step 1 Find the companion matrices $\tilde{A}$ and $\tilde{B}$. Compute $W = T\tilde{T}^{-1}$.

Step 2 Apply the control law $u = -\tilde{K}x$ to the companion system $(\tilde{A}, \tilde{B})$, given in Equation 1.11 and Equation 1.12,

$$\tilde{A} - \tilde{B}\tilde{K} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_1 - k_1 & -a_2 - k_2 & \cdots & -a_{n-1} - k_{n-1} & -a_n - k_n \end{bmatrix}. \tag{1.13}$$

Compute the control gain, $\tilde{K}$, that assigns the poles of $\tilde{A} - \tilde{B}\tilde{K}$ to the desired locations.

Step 3 Find $K = \tilde{K}W^{-1}$ to get the feedback gain for the original system (A, B).

Remark: It is important to do the $\tilde{K} \to K$ conversion. Remember that (A, B) represents the *actual* system while the companion matrices $\tilde{A}$ and $\tilde{B}$ do not.

1.4 Time-Domain Specifications for Higher Order Systems

The rotary inverted pendulum system has four poles. However, if two of the closed-loop poles are chosen to be closer to the imaginary axis (typically by a factor equal or greater than four) than the remaining poles, the conjugate poles are considered to be *dominant* and the system's behavior can be approximated by a second-order system. As depicted in Figure 1.2, poles p_1 and p_2 are the complex conjugate *dominant* poles and are chosen to satisfy the natural frequency, ω_n, and damping ratio, ζ, second-order specifications. Let the conjugate poles be

$$p_1 = -\sigma + j\omega_d \tag{1.14}$$

and

$$p_2 = -\sigma - j\omega_d \tag{1.15}$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the *damped* natural frequency. The remaining closed-loop poles, p_3 and p_4, are placed along the real-axis to the left of the dominant poles, as shown in Figure 1.2.

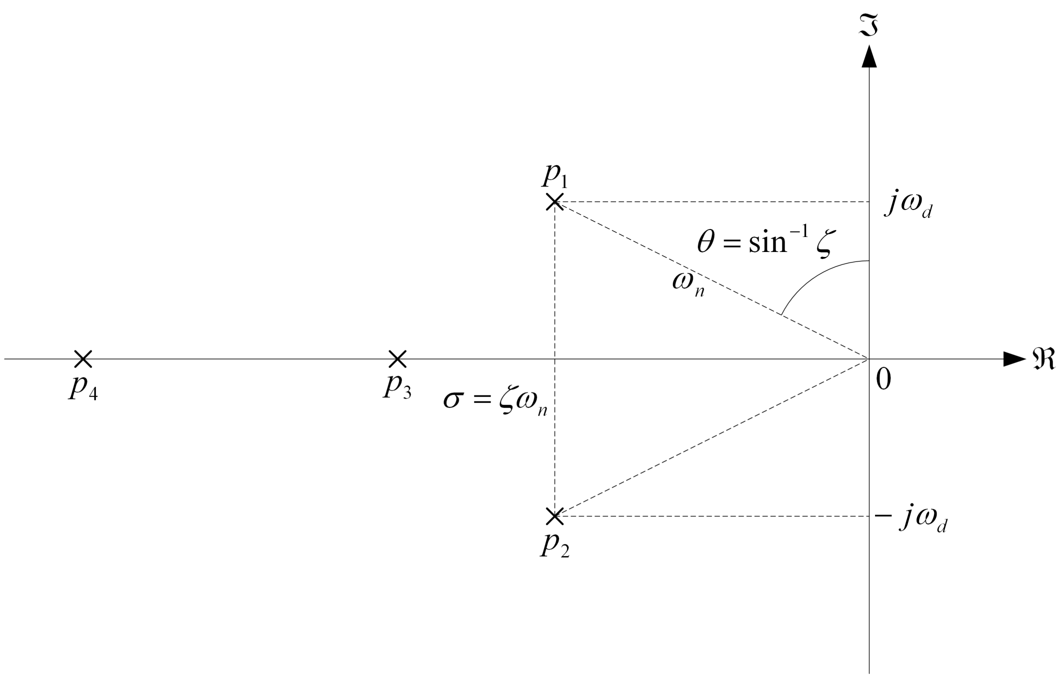


Figure 1.2: Desired closed-loop pole locations

2 In-Lab Exercises

2.1 Pole Placement Control Design

The time-domain specifications when the pendulum is balanced and tracking a rotary arm setpoint are:

$$\begin{aligned} PO &\leq 6.81\% \\ t_s &\leq 1.54 \text{ s} \end{aligned} \tag{2.1}$$

Thus, as the rotary arm goes back and forth to track the reference (while balancing the pendulum) it should have a percent overshoot and settling time matching these requirements.

1. The open-loop poles of the inverted pendulum are located at $\{0, -22.03, 13.58, -4.672\}$. Note that the system is unstable. Find the characteristic equation of A.
2. Find the companion matrices $(\tilde{A}, \tilde{B})$.
3. The percent overshoot and settling time specifications given in Equation 2.1 translates into the following natural frequency and damping ratio requirements:

   $$\begin{aligned} \zeta &= 0.65 \\ \omega_n &= 4 \text{ rad/s} \end{aligned} \tag{2.2}$$

   Based on these specifications, find the location of the two dominant poles p_1 and p_2.
4. Find the desired characteristic equation if the other poles are placed at $p_3 = -40$ and $p_4 = -45$.
5. Applying the companion-based control $u = -\tilde{K}x$ to $(\tilde{A}, \tilde{B})$ results in the closed-loop system $(\tilde{A} - \tilde{K}\tilde{B})$. Find the gain $\tilde{K}$ that assigns the poles to their new desired location.
6. Open the MATLAB® software.
7. Design a MATLAB script that evaluates the controllability of the Rotary Inverted Pendulum system. Use the state-space model defined in the `rotpen_ABCD_eqns_ip.m` and the model parameters in the `qube2_rotpen_param.m` scripts supplied. Attach the resulting controllability matrix and explain whether or not the system is controllable.

   Hint: The MATLAB® *Control System Toolbox* includes a command to find the controllability matrix. Look through the documentation to find the appropriate command.
8. Use the pole-placement design method in MATLAB® to find the state-feedback gain, K, required to place the closed-loop poles to the desired poles. Show the MATLAB commands used and the obtained gain.

   Hint: The pole-placement design algorithm has been packaged in the MATLAB® *Control System Toolbox*. Look through the documentation to find the appropriate command.

2.2 Pole Placement-Based Balance Control

Construct a QUARC® controller similarly to Figure 2.1 that balances the pendulum on the QUBE-Servo 2 rotary pendulum system using a generated control gain K.

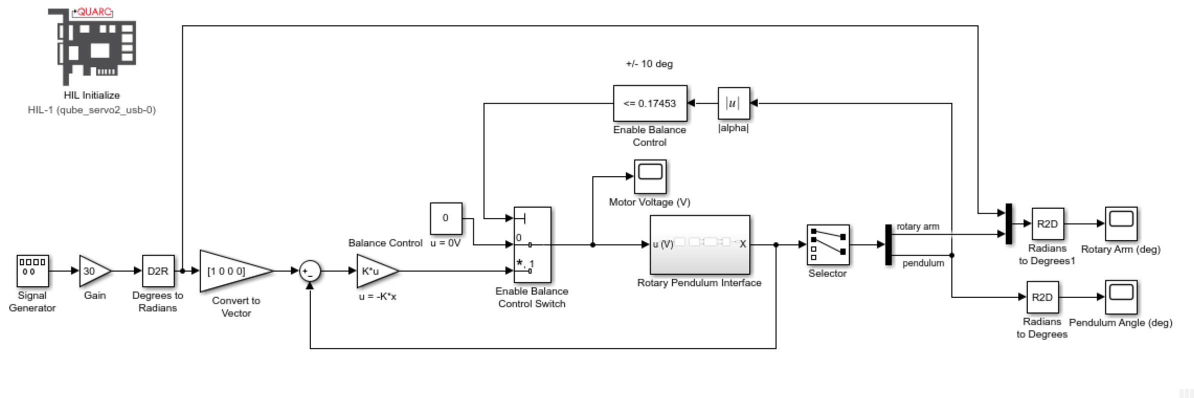


Figure 2.1: SIMULINK®/QUARC® model implementing the rotary pendulum balance controller

1. If supplied, open the `q_qube2_rotpen_pp` SIMULINK® model. Otherwise, use the SIMULINK® model you made in the Rotary Pendulum Modeling laboratory experiment to construct the controller shown in Figure 2.1:

   - Using the angles from the Counts to Angles subsystem you designed in the Rotary Pendulum Modeling laboratory experiment (which converts encoder counts to radians), build state x given in Equation 1.5. In Figure 2.1, it is bundled in the subsystem called *State X*. Use high-pass filters $50s/(s+50)$ to compute the velocities $\dot{\theta}$ and $\dot{\alpha}$.
   - Add the necessary Sum and Gain blocks to implement the state-feedback control given in Equation 1.7. Since the control gain is a vector, make sure the gain block is configured to do matrix type multiplication.
   - Add the Signal Generator block in order to generate a varying, desired arm angle. To generate a reference state, make sure you include a Gain block of $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$.

2. Load the state-feedback gain K you found in Section 2.1. Make sure it is set as variable `K` in the MATLAB® workspace. Alternatively, if supplied, run the `setup_qube2_pp.m` script in MATLAB.

3. Set the Signal Generator block to the following:

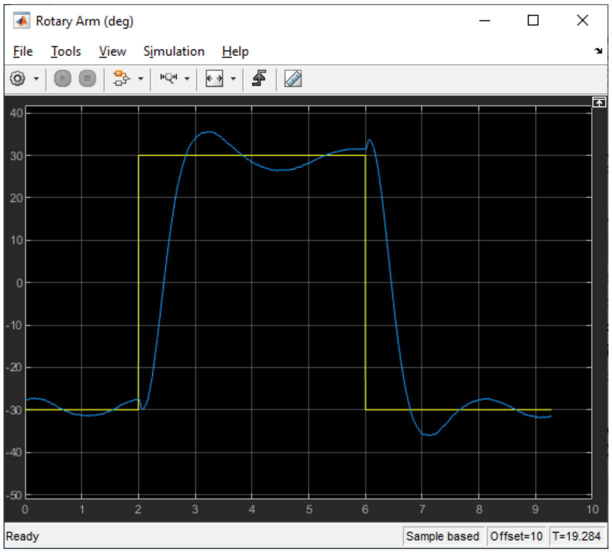
   - Type = Square
   - Amplitude = 1
   - Frequency = 0.125 Hz

4. Set the Gain block that is connected to the Signal Generator to 0.

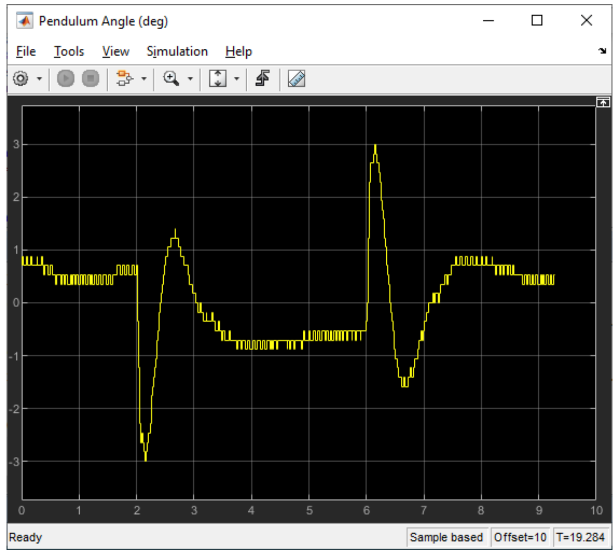
5. Build and run the QUARC® controller.

6. Manually rotate the pendulum in the upright position until the controller engages.

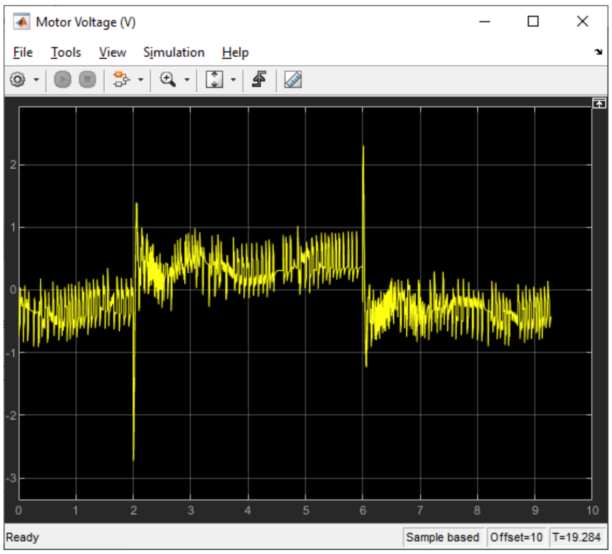
7. Once the pendulum is balanced, set the Gain to 30 to make the arm angle go between $\pm 30°$. The scopes shown in Figure 2.2 show an *example response* when using a state-feedback gain of $K = \begin{bmatrix} -2 & 35 & -1 & 3 \end{bmatrix}$. Attach the response of the rotary arm, pendulum, and controller voltage using the control gain found above in Section 2.1.



(a) Rotary Arm



(b) Pendulum



(c) Motor Voltage

Figure 2.2: QUBE-Servo 2 example rotary pendulum response using default gain

8. Does the rotary arm and pendulum response match the settling time and percent overshoot specifications given in Equation 2.1? If not, give one reason why there is a discrepancy.
9. Stop the QUARC® controller.
10. Power off the QUBE-Servo 2 if no more experiments will be conducted.

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BIBLIOGRAPHY

[1] Chi-Tsong Chen. *Linear System Theory and Design*. Oxford University Press, 1999.