Aero 2 - 2 DOF Helicopter Modeling

Concept Review

The free-body diagram of the Aero 2 system is illustrated in Figure 1.

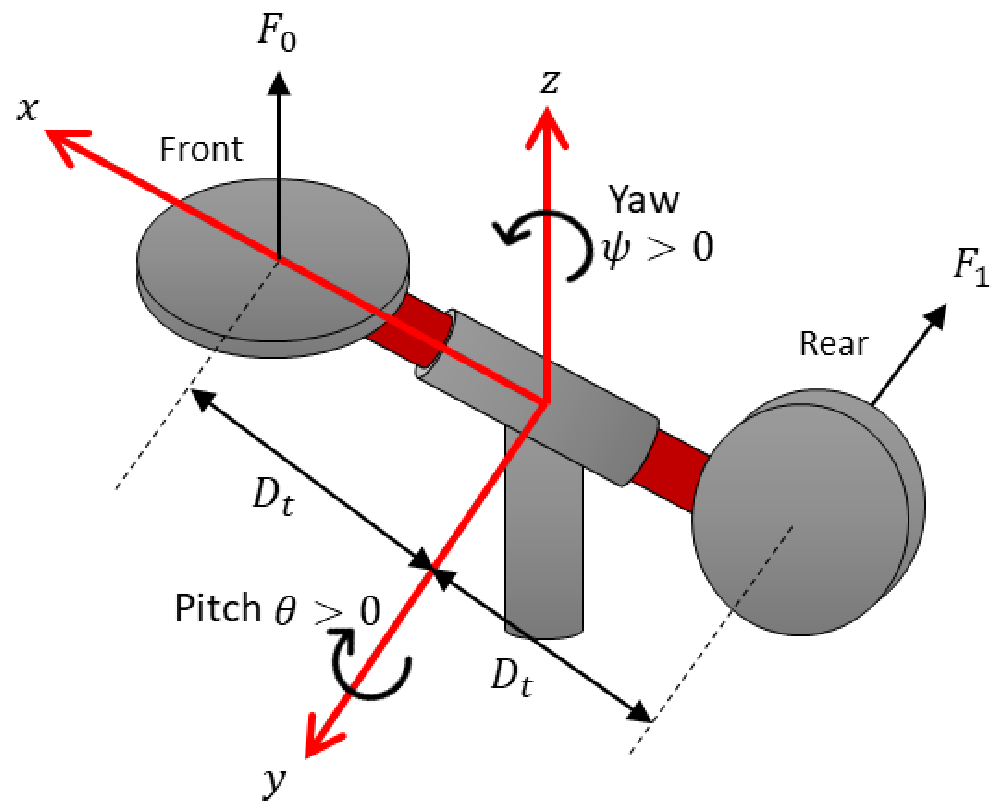

Figure 1: Free-body diagram of Aero 2 Experiment

The following conventions are used for the modeling:

1. The helicopter is horizontal and parallel with the ground when the pitch angle is zero, $\theta = 0$.
2. The pitch angle increases positively, $\dot{\theta}(t) > 0$, when the body rotates **clockwise (CCW)** about the y-axis, i.e., when the front rotor goes up.
3. The yaw angle increases positively, $\dot{\psi}(t) > 0$, when the body rotates **counter-clockwise (CCW)** about the z-axis.
4. Pitch increases, $\dot{\theta} > 0$, when the front/pitch rotor voltage is positive, $V_p > 0$.
5. Yaw increases, $\dot{\psi} > 0$, when the rear/yaw rotor voltage is positive, $V_y > 0$.

When a voltage is applied to the pitch motor, V_p, the front rotor generates a force, F_0, that acts normal to the body at a distance D_t from the pitch axis (along the x-axis). As discussed in the *Aero2 - 2 DOF Helicopter - Introduction* document, due to aerodynamic drag the torque generated from the rotation of the front propeller blade also generates a torque about the yaw-axis (z-axis). That's why conventional helicopters include a tail, or anti-torque, rotor to compensate for the torque generated about the yaw axis by the large, main rotor. Similarly, the rear motor generates a force F_1 that acts on the body at a distance D_t from the yaw axis. This generates a torque about the yaw axis and a torque about the pitch axis.

Based on this, we can represent the motions of the Aero 2 about the horizontal (i.e., when the body is parallel with the ground) using the following equations of motion (EOMs):

$$
\begin{aligned}
J_p\ddot{\theta} + D_p\dot{\theta} + K_{sp}\theta &= \tau_p, \qquad (1) \\
J_y\ddot{\psi} + D_y\dot{\psi} &= \tau_y \qquad (2)
\end{aligned}
$$

where the torques acting on the pitch and yaw axes are

$$
\begin{aligned}
\tau_p &= K_{pp}D_tV_p + K_{py}D_tV_y, \text{ and} \\
\tau_y &= K_{yp}D_tV_p + K_{yy}D_tV_y.
\end{aligned}
$$

The parameters used in the EOMs above are:

- J_p is the total moment of inertia about the pitch axis,
- J_y is the total moment of inertia about the yaw axis,
- D_p is the damping about the pitch axis,
- D_y is the damping about the yaw axis,
- K_{sp} is the stiffness about the pitch axis,
- K_{pp} is the thrust force gain acting on the pitch axis from the pitch/front rotor,
- K_{py} is the thrust force gain acting on the pitch from the yaw/rear rotor,
- K_{yy} is the thrust force gain acting on the yaw axis from the yaw/rear rotor,
- K_{yp} is the thrust force gain acting on the yaw axis from the pitch/front rotor,
- D_t is the distance between the Aero 2 pivot and center of the rotor,
- V_p is the voltage applied to the front pitch rotor, and
- V_y is the voltage applied to the rear yaw rotor motor.

Some of these model parameters are given in the Aero 2 User Manual. The rest are found experimentally.

Note: This model is not exhaustive in that it does not take into account all of the system dynamics, e.g., nonlinearities. It is meant to be used for designing linear control systems. However, it does capture the gyroscopic reaction torques that act on each axis from both rotors.

Transfer Function Model

Taking the Laplace transform of the equations of motion

$$
J_p\left(\Theta(s)s^2 - \theta(0^-)s - \dot{\theta}(0^-)\right) + D_p(\Theta(s)s - \theta(0^-)) + K_{sp}\Theta(s) = K_{pp}D_tV_p(s) + K_{py}D_tV_y(s)
$$

and

$$J_y(\Psi(s)s^2 - \psi(0^-)s - \dot{\psi}(0^-)) + D_y(\Psi(s)s - \psi(0^-)) = K_{yp}D_tV_p(s) + K_{yy}D_tV_y(s)$$

This is a multiple-input, multiple-output (MIMO) system with two outputs and two inputs. Since the system starts when it is at rest, all the initial conditions are zero, i.e., $\theta(0^-) = 0$, $\dot{\theta}(0^-) = 0$ $\psi(0^-) = 0$, and $\dot{\psi}(0^-) = 0$. Given this we can obtain the following transfer functions describing the system motions relative to the different inputs:

$$\Theta(s) = \frac{K_{pp}D_t}{J_ps^2 + D_ps + K_{sp}}V_p(s) + \frac{K_{py}D_t}{J_ps^2 + D_ps + K_{sp}}V_y(s) \qquad (3)$$

and

$$\Psi(s) = \frac{K_{yp}D_t}{J_ys^2 + D_ys}V_p(s) + \frac{K_{yy}D_t}{J_ys^2 + D_ys}V_y(s). \qquad (4)$$

Linear State-Space Representation

The linear state-space equations are:

$$\dot{x} = Ax + Bu \text{ and } y = Cx + Du.$$

Using the equations of motion, we can define the following state-space matrices for the Aero 2 system

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K_{sp}/J_p & 0 & -D_p/J_p & 0 \\ 0 & 0 & 0 & -D_y/J_y \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ D_tK_{pp}/J_p & D_tK_{py}/J_p \\ D_tK_{yp}/J_y & D_tK_{yy}/J_y \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$x^T = \left[\theta(t), \psi(t), \dot{\theta}(t), \dot{\psi}(t)\right], \qquad (5)$$

is the system state,

$$y^\top = [\theta(t)\ \psi(t)]$$

is the output state, and

$$u^\top = [V_p(t)\ V_y(t)]$$

is the control input.